A Drug-Induced Random Walk

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Abstract

The label on a bottle of pills says "Take one half pill daily." A natural way to proceed is as follows: Every day, remove a pill from the bottle at random. If it is a whole pill, break it in half, take one half, and return the other half to the bottle; if it is a half pill, take it. We analyze the history of such a pill bottle.

1 Introduction

A few years ago our cat Natasha (see Figure 1) began losing weight. We took her to the vet, who did some tests and determined that she had a thyroid condition. He gave us a bottle of pills and told us to give her half a pill every day.

The next day we shook a pill out of the bottle, broke it in half, gave her half of the pill, and put the other half back in the bottle. We repeated that procedure for several more days. Eventually, a day came when the pill we shook out of the bottle was one of the half pills we had put back in on one of the previous days. Of course, we just gave her the half pill that day. We continued to follow this procedure until the bottle was empty, and then we started on a new bottle.

The pills solved Natasha's medical problem; she regained the weight she had lost, and she's doing fine now. But they created an interesting mathematical problem. The state of the pill bottle on any day can be described by a pair of numbers (w, h), where w is the number of whole pills in the bottle and h is the number of half pills. We will assume that

Figure 1: Natasha.

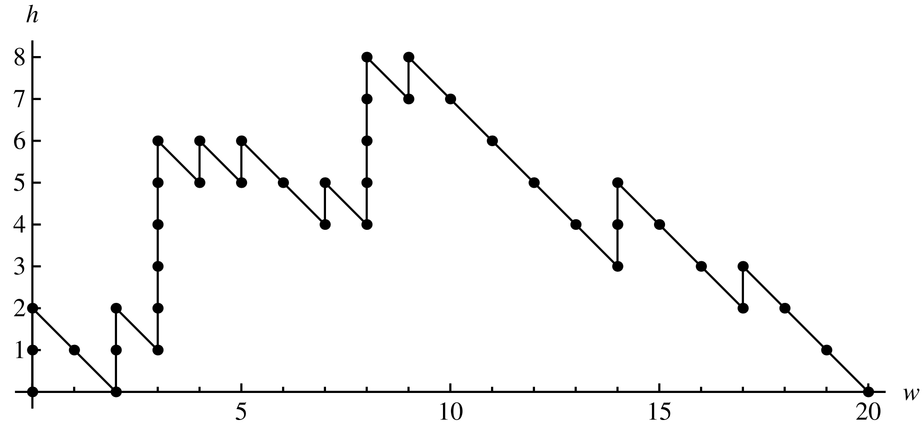

Figure 2: A pill-bottle walk with $n = 20$.

every day a pill is removed from the bottle at random, with each pill being equally likely to be chosen. When a whole pill is removed, it is cut in half and half of it is returned to the bottle; when a half pill is removed, nothing is returned to the bottle. Thus, if the state of the pill bottle on a particular day is (w, h), then with probability $w/(w+h)$ the state on the next day will be $(w-1, h+1)$, and with probability $h/(w+h)$ it will be $(w, h-1)$. This means that the state of the pill bottle executes a random walk in the plane, starting at the point $(w, h) = (n, 0)$, where n is the initial number of pills in the bottle, and ending at $(0, 0)$. Since the bottle contains $2n$ doses of medicine, the walk takes $2n$ steps.

For example, Figure 2 shows a computer simulation of a pill-bottle walk starting with $n = 20$ pills. On the first three days, whole pills are removed from the bottle, and the state of the bottle goes from $(20, 0)$ to $(19, 1)$, $(18, 2)$, and $(17, 3)$. The next day, a half pill is removed, and the state goes to $(17, 2)$. And the walk continues for 36 more steps until it ends at $(0, 0)$.

Figure 3 shows simulated walks with $n = 100$, $n = 1000$, and $n = 10000$. It appears that although the walks are random, the overall shapes of the walks are similar, with the shape becoming smoother as n increases. Notice that the scales of the three walks in Figure 3 are different; the first starts at $(100, 0)$, the second at $(1000, 0)$, and the third at $(10000, 0)$. It is only when they are drawn the same size that they look similar. This suggests that we should rescale the walks to a uniform size, independent of n. We will therefore switch to a new coordinate system. If we let $x = w/n$ and $y = h/n$, then x represents the fraction of the original n pills that are still whole, and y represents the fraction that have become half pills. Notice that these fractions may add up to less than 1, since some fraction of the pills may have been used up completely.

Using the coordinates (x, y) to represent the state of the pill bottle, we get a random walk that starts at $(1, 0)$, ends at $(0, 0)$, and stays in the triangle $x + y \le 1$, $x \ge 0$, $y \ge 0$. When the state is (x, y), it changes as follows:

- with probability $\dfrac{x}{x+y}$, the state changes to $\left(x - \dfrac{1}{n}, y + \dfrac{1}{n}\right)$;
- with probability $\dfrac{y}{x+y}$, the state changes to $\left(x, y - \dfrac{1}{n}\right)$.

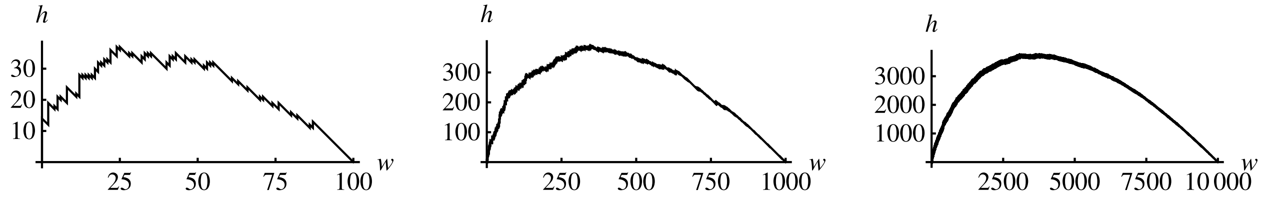

Figure 3: Walks with $n = 100$ (left), $n = 1000$ (center), and $n = 10000$ (right).

We will call such a walk an n-*walk*. Increasing n does not make the walk larger, but it makes the steps smaller. Figure 3 suggests that as n increases, the walk approaches a smooth curve. What is this curve?

The limit curve we seek is an example of a *scaling limit* of a discrete process. Perhaps the best-known example of a scaling limit is Brownian motion, which can also be thought of as the scaling limit of a random walk. For more on Brownian motion and scaling limits, see [5].

We first give an intuitive argument that suggests a possible answer to our question. We will find it helpful to introduce a third variable t, standing for time. We set $t = 0$ at the beginning of the walk, and to keep the scales of the variables comparable we will assume that t increases by $1/n$ for each step of the walk. Since the walk consists of $2n$ steps, this means that t will run from 0 to 2. We think of the limit curve as being given by parametric equations

$$x = f_x(t), \qquad y = f_y(t), \qquad 0 \le t \le 2,$$

or, in vector notation,

$$(x, y) = (f_x(t), f_y(t)) = \mathbf{f}(t), \qquad 0 \le t \le 2.$$

When the state of an n-walk is (x, y), the displacement to the next state is either the vector $(-1/n, 1/n)$, with probability $x/(x + y)$, or $(0, -1/n)$, with probability $y/(x + y)$. Thus, the expected value of the displacement is

$$\frac{x}{x+y}\left(-\frac{1}{n}, \frac{1}{n}\right) + \frac{y}{x+y}\left(0, -\frac{1}{n}\right) = \frac{1}{n}\left(-\frac{x}{x+y}, \frac{x-y}{x+y}\right).$$

Since t increases by $1/n$ during the step, this suggests that the parametric form of the limit curve might be a solution to the system of differential equations

$$\frac{dx}{dt} = -\frac{x}{x+y}, \qquad \frac{dy}{dt} = \frac{x-y}{x+y}. \tag{1}$$

To solve this system of equations, we first note that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{x-y}{x} = -1 + \frac{y}{x}.$$

We will let you check that the curve $y = -x \ln x$ satisfies this equation for $0 < x \le 1$ and passes through the point $(1, 0)$. The graph of this curve is shown in Figure 4, and

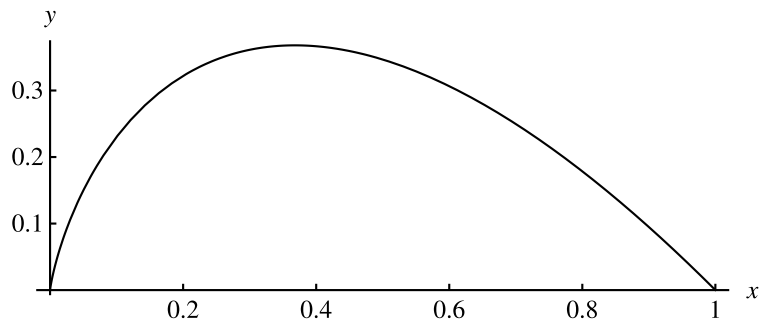

Figure 4: The graph of $y = -x \ln x$.

the similarity to the walks in Figure 3 is striking. Notice that although $\ln 0$ is undefined, $\lim_{x \to 0^+}(x \ln x) = 0$. From now on we consider $0 \ln 0$ to be equal to 0, so that the curve $y = -x \ln x$ includes the point $(0, 0)$.

Substituting $y = -x \ln x$ in the first equation in (1), we get

$$\frac{dx}{dt} = -\frac{x}{x - x \ln x} = \frac{1}{\ln x - 1}.$$

Separation of variables gives

$$t = \int (\ln x - 1)\, dx = x \ln x - 2x + C.$$

Since $x = 1$ when $t = 0$, we must have $C = 2$, and therefore

$$t = x \ln x - 2x + 2. \tag{2}$$

Let $g(x) = x \ln x - 2x + 2$ for $0 \leq x \leq 1$. (Notice that by our convention that $0 \ln 0 = 0$, we have $g(0) = 2$.) Then g maps $[0, 1]$ onto $[0, 2]$ and is strictly decreasing, so it has an inverse. We define f_x to be the inverse of g, which is a strictly decreasing function mapping $[0, 2]$ to $[0, 1]$. Thus, if $0 \leq t \leq 2$ and $x = f_x(t)$, then x and t satisfy equation (2).[1]

Using $y = -x \ln x$, we can rewrite equation (2) as $t = -y - 2x + 2$, or equivalently $y = 2 - 2x - t$. We therefore define

$$f_y(t) = 2 - 2f_x(t) - t. \tag{3}$$

We leave it to you to verify that the equation

$$(x, y) = (f_x(t), f_y(t)) = \mathbf{f}(t), \qquad 0 \leq t \leq 2 \tag{4}$$

parametrizes the curve $y = -x \ln x$ shown in Figure 4, and it satisfies the differential equations (1) for $0 \leq t < 2$, where we interpret the derivatives at $t = 0$ as one-sided derivatives. (At $t = 2$, we have $x = y = 0$, and therefore the right-hand sides of the equations in (1) are undefined.) The graphs of f_x and f_y are shown in Figure 5.

[1] Using the Lambert W function W_{-1} (see [1]), we can express $f_x(t)$ explicitly by the equation

$$f_x(t) = \frac{t - 2}{W_{-1}((t - 2)/e^2)}.$$

However, we will not have any use for this expression.

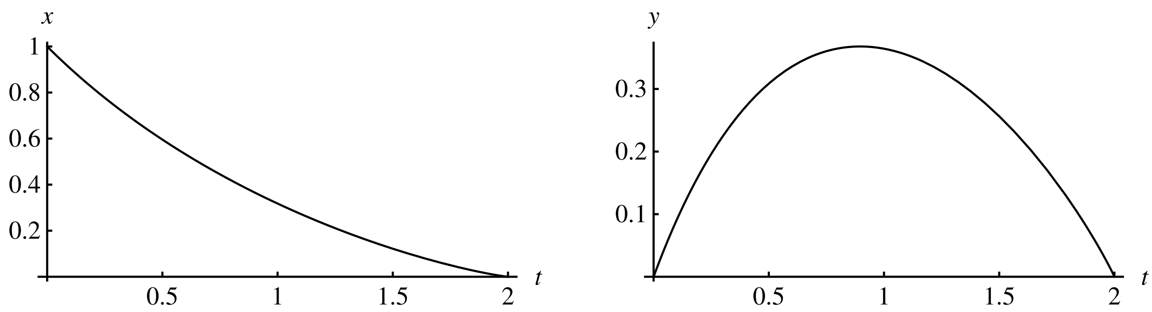

Figure 5: The graphs of $x = f_x(t)$ (left) and $y = f_y(t)$ (right).

It turns out that an n-walk does, indeed, approach the curve (4) as n approaches ∞, but the sense in which this is true must be stated carefully. Our main theorem is the following.

Theorem 1. *Suppose $\epsilon > 0$. Let the points on an n-walk be $\mathbf{p}_0 = (1, 0)$, $\mathbf{p}_1, \ldots, \mathbf{p}_{2n} = (0, 0)$, and for $0 \leq i \leq 2n$ let $t_i = i/n$. Then the probability that for every i, $\|\mathbf{p}_i - \mathbf{f}(t_i)\| < \epsilon$ approaches 1 as $n \to \infty$. In other words, the n-walk converges uniformly in probability to the limit curve.*

Two notable features of the limit curve are that the tangent line at $(1, 0)$ has slope -1, and the tangent line at the origin is vertical. The first feature makes intuitive sense: early in the walk, almost all of the pills in the bottle are whole pills, so it is likely that several whole pills will be removed before the first half pill is removed. For example, in the walk in Figure 2, three whole pills were removed before the first half pill was removed. When these initial whole pills are removed, the walk will move along the line $y = 1 - x$, which is the tangent line at $(1, 0)$. The second feature seems more surprising: it appears that near the end of the walk, almost all of the pills are half pills, and the walk ends by moving along the line $x = 0$ toward the origin. This suggests two questions.

Question 1. For a bottle of n pills, what is the expected number of whole pills that are removed from the bottle before the first half pill is removed?

Question 2. For a bottle of n pills, what is the expected number of half pills that are removed from the bottle after the last whole pill is removed?

Versions of Question 1 have appeared in the literature before (see, for example, [3, 4, 6, 8]). In the case $n = 365$, it is equivalent to the following version of the birthday problem: If people are chosen at random, one by one, what is the expected number of people with distinct birthdays who will be chosen before the first person who has the same birthday as a previously chosen person? We will give an elementary derivation of the answer to Question 1. In our next theorem we express the answer in terms of the *incomplete gamma function*, which is defined as follows:

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} \, dt.$$

Theorem 2. *For a bottle of n pills, the expected number of whole pills that are removed from the bottle before the first half pill is removed is*

$$\frac{e^n}{n^{n-1}}\Gamma(n,n).$$

As $n \to \infty$, this expected value is asymptotic to

$$\sqrt{\frac{\pi n}{2}}.$$

The answer to Question 2 was found by Richard Stong.

Theorem 3 (Stong)**.** *For a bottle of n pills, the expected number of half pills that are removed from the bottle after the last whole pill is removed is the nth harmonic number,*

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For example, for a bottle of 100 pills, the expected number of whole pills before the first half pill is

$$\frac{e^{100}}{100^{99}}\Gamma(100,100) \approx 12.21,$$

and the asymptotic approximation in Theorem 2 is

$$\sqrt{\frac{100\pi}{2}} \approx 12.53.$$

The expected number of half pills after the last whole pill is

$$H_{100} \approx 5.19.$$

The rest of this paper is devoted to the proofs of Theorems 1–3. We prove Theorem 1 in Section 3, and Theorems 2 and 3 in Section 4. We consider variations on these theorems in Section 5.

2 Background for Proof of Theorem 1

In preparation for the proof of Theorem 1, we simplify the problem by eliminating one variable. According to definition (3), $f_y(t) = 2 - 2f_x(t) - t$, so

$$\mathbf{f}(t) = (f_x(t), 2 - 2f_x(t) - t) = f_x(t)(1, -2) + (0, 2 - t).$$

A similar equation holds for the points on any n-walk. Suppose that after i steps, the n-walk is at the point $\mathbf{p}_i = (x_i, y_i)$, and let $t_i = i/n$. This means that there are $w_i = nx_i$ whole pills and $h_i = ny_i$ half pills in the bottle. These pills are enough for $2w_i + h_i$ doses of medicine. Since there were $2n$ doses in the bottle originally, and i of those doses have been used up, there must be $2n - i$ doses left. Therefore $2w_i + h_i = 2n - i$, or equivalently $h_i = 2n - 2w_i - i$. Dividing through by n, we find that

$$y_i = 2 - 2x_i - t_i, \tag{5}$$

and therefore

$$\mathbf{p}_i = (x_i, 2 - 2x_i - t_i) = x_i(1, -2) + (0, 2 - t_i).$$

It follows that

$$\|\mathbf{p}_i - \mathbf{f}(t_i)\| = \|(x_i - f_x(t_i))(1, -2)\| = |x_i - f_x(t_i)|\sqrt{5}.$$

Thus, to ensure that $\mathbf{p}_i$ is close to $\mathbf{f}(t_i)$, it will suffice to ensure that x_i is close to $f_x(t_i)$; we can ignore the y-coordinates of $\mathbf{p}_i$ and $\mathbf{f}(t_i)$. In other words, to prove Theorem 1 it will suffice to prove the following lemma.

Lemma 4. *Suppose $\epsilon > 0$. Let the x-coordinates of the points on an n-walk be $x_0 = 1$, $x_1, \ldots, x_{2n} = 0$, and for $0 \le i \le 2n$ let $t_i = i/n$. Then the probability that for every i, $|x_i - f_x(t_i)| < \epsilon$ approaches 1 as $n \to \infty$.*

In fact, using equations (3) and (5) we can completely eliminate the variable y from the problem. We can describe the x-coordinates of the points on an n-walk by saying that x_{i+1} is equal to either $x_i - 1/n$ or x_i, with the first possibility occurring with probability

$$\frac{x_i}{x_i + y_i} = \frac{x_i}{x_i + 2 - 2x_i - t_i} = \frac{x_i}{2 - x_i - t_i}. \tag{6}$$

Similarly, if $x = f_x(t)$ and $y = f_y(t)$, then for $0 \le t < 2$,

$$f_x'(t) = \frac{dx}{dt} = -\frac{x}{x + y} = -\frac{x}{2 - x - t} = -\frac{f_x(t)}{2 - f_x(t) - t}. \tag{7}$$

Thus, we can work entirely with the points (t_i, x_i) and the curve $x = f_x(t)$, both of which lie in the tx-plane.

The idea behind our proof of Lemma 4 is straightforward. Let m be a large positive integer, and let n be an integer much larger than m. Now consider an n-walk, and break the $2n$ steps of the walk into m large blocks of steps. We view the n-walk in the tx-plane, ignoring the y-coordinates. The individual steps of the n-walk are random and unpredictable, but the net change in x that results from a large block of steps is more predictable: by the law of large numbers, this net change is likely to be close to its expected value. It will follow that if a block of steps starts at a point (t, x), then the net result of this block of steps is likely to be a small displacement in the tx-plane whose slope is close to $-x/(2-x-t)$. Since $x = f_x(t)$ is a solution to the differential equation $dx/dt = -x/(2 - x - t)$, this means that the steps of the n-walk should stay close to the graph of f_x.

This proof sketch suggests that our proof will involve ideas related to Euler's method. Recall that Euler's method is a numerical method for solving a differential equation of the form $f'(t) = F(t, f(t))$ for $a \le t \le b$, with an initial condition $f(a) = x_0$. Here the function F and the numbers a, b, and x_0 are given, and we want to compute values of f. To apply

Euler's method, we choose a positive integer n and a positive step size $h \leq (b-a)/n$, let $t_j = a + jh$ for $0 \leq j \leq n$, and then define x_j recursively by the equation

$$x_{j+1} = x_j + hF(t_j, x_j), \quad 0 \leq j < n.$$

Thus, the displacement from (t_j, x_j) to (t_{j+1}, x_{j+1}) has slope $F(t_j, x_j)$. If h is small and F is sufficiently well behaved, then the points (t_j, x_j) will be close to the graph of f.

We will need to modify Euler's method slightly, because according to our proof sketch for Lemma 4, the slope of the displacement caused by a block of steps in the n-walk starting at (t, x) is likely to be close to $-x/(2-x-t)$, but not exactly equal to it. We will therefore need a version of Euler's method in which the slope of the displacement at step j is only approximately equal to $F(t_j, x_j)$.

To make this precise, suppose that $a < b$, g_1 and g_2 are functions from $[a, b]$ to $\mathbb{R}$, and for all $t \in [a, b]$, $g_1(t) < g_2(t)$. Let

$$D = \{(t, x) \in \mathbb{R}^2 : a \leq t \leq b \text{ and } g_1(t) \leq x \leq g_2(t)\}.$$

Now suppose that $F : D \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$, and for all $t \in [a, b]$, $(t, f(t)) \in D$ and

$$f'(t) = F(t, f(t)),$$

where we interpret $f'(t)$ as a one-sided derivative when $t = a$ or $t = b$. Let $x_0 = f(a)$. We want to use a version of Euler's method to locate points (t_j, x_j) near the graph of f. As before, we will use a positive step size $h \leq (b-a)/n$, so for $0 \leq j \leq n$ we let $t_j = a + jh$. We will assume that for $0 \leq j < n$, the slope of the displacement from (t_j, x_j) to (t_{j+1}, x_{j+1}) deviates from $F(t_j, x_j)$ by some amount δ_j. Thus, we recursively define

$$x_{j+1} = x_j + h(F(t_j, x_j) + \delta_j).$$

To ensure that this formula is defined, we assume that for every j, $g_1(t_j) \leq x_j \leq g_2(t_j)$, so that $(t_j, x_j) \in D$.

Lemma 5. *In the modified Euler's method described above, assume that for* $0 \leq j < n$,

$$|\delta_j| \leq \delta.$$

We also assume that $\partial F/\partial x$ *and* f'' *are defined and bounded. Thus, we assume that there are positive constants* C_1 *and* C_2 *such that for all* $(t, x) \in D$,

$$\left| \frac{\partial F}{\partial x}(t, x) \right| \leq C_1, \qquad |f''(t)| \leq C_2.$$

Then for $0 \leq j \leq n$,

$$|x_j - f(t_j)| \leq \left(\frac{hC_2}{2C_1} + \frac{\delta}{C_1} \right) \left((1 + C_1 h)^j - 1 \right). \tag{8}$$

Proof. We proceed by induction on j. Clearly inequality (8) holds when $j = 0$, since both sides are 0. Now suppose the inequality holds for some $j < n$. By Taylor's theorem, we can write

$$f(t_{j+1}) = f(t_j) + hf'(t_j) + \frac{h^2}{2} f''(c_j)$$

for some number c_j between t_j and t_{j+1}. And by the mean value theorem, we have

$$F(t_j, x_j) = F(t_j, f(t_j)) + \frac{\partial F}{\partial x}(t_j, d_j)(x_j - f(t_j)) = f'(t_j) + \frac{\partial F}{\partial x}(t_j, d_j)(x_j - f(t_j))$$

for some d_j between x_j and $f(t_j)$. Thus,

$$\begin{aligned}
x_{j+1} - f(t_{j+1}) &= x_j + h(F(t_j, x_j) + \delta_j) - f(t_{j+1}) \\
&= x_j + h\left(f'(t_j) + \frac{\partial F}{\partial x}(t_j, d_j)(x_j - f(t_j)) + \delta_j\right) \\
&\qquad\qquad - \left(f(t_j) + hf'(t_j) + \frac{h^2}{2} f''(c_j)\right) \\
&= (x_j - f(t_j))\left(1 + h\frac{\partial F}{\partial x}(t_j, d_j)\right) + h\delta_j - \frac{h^2}{2} f''(c_j).
\end{aligned}$$

Next we take absolute values and apply the bounds given in the statement of the lemma:

$$|x_{j+1} - f(t_{j+1})| \le |x_j - f(t_j)|(1 + C_1 h) + h\delta + \frac{C_2 h^2}{2}.$$

Finally, we apply the inductive hypothesis to conclude that

$$\begin{aligned}
|x_{j+1} - f(t_{j+1})| &\le \left(\frac{hC_2}{2C_1} + \frac{\delta}{C_1}\right)\left((1 + C_1 h)^j - 1\right)(1 + C_1 h) + h\delta + \frac{C_2 h^2}{2} \\
&= \left(\frac{hC_2}{2C_1} + \frac{\delta}{C_1}\right)\left((1 + C_1 h)^{j+1} - 1\right),
\end{aligned}$$

as required. □

3 Proof of Theorem 1

To complete the proof of Theorem 1, we return to our proof sketch for Lemma 4. Unfortunately, nailing down the details of this proof sketch is not easy. Nevertheless, in this section we show that, with some care, a proof based on these ideas can be carried out.

Fix $\epsilon > 0$. We will refer to the region $f_x(t) - \epsilon < x < f_x(t) + \epsilon$ in the tx-plane as the *ϵ-corridor*. To prove Lemma 4, we must show that for large n, an n-walk is likely to stay entirely inside the ϵ-corridor. We first determine simple bounds on any n-walk. At step i of the walk, by (5) we have

$$x_i \ge 0, \qquad 2 - 2x_i - t_i = y_i \ge 0,$$

and therefore

$$0 \le x_i \le \frac{2 - t_i}{2}. \tag{9}$$

Similar bounds apply to the graph of f_x: for $0 \le t \le 2$,

$$0 \le f_x(t) \le 1, \qquad 2 - 2f_x(t) - t = f_y(t) = -f_x(t)\ln(f_x(t)) \ge 0,$$

so

$$0 \le f_x(t) \le \frac{2 - t}{2}. \tag{10}$$

These simple bounds already imply that the end of the n-walk stays inside the ϵ-corridor: if $t_i > 2 - 2\epsilon$, then

$$0 \le x_i, f_x(t_i) \le \frac{2 - t_i}{2} < \epsilon,$$

and therefore

$$|x_i - f_x(t_i)| < \epsilon.$$

Thus, we only need to worry about t_i in the interval $[0, 2 - 2\epsilon]$. In particular, if $\epsilon > 1$, then there is nothing more to prove, so we can assume now that $\epsilon \le 1$. By stopping short of $t = 2$, we avoid having to deal with the point $(t, x, y) = (2, 0, 0)$ on the limit curve, where the right-hand sides of the equations in (1) are undefined.

We will find it convenient to go a bit beyond $t = 2 - 2\epsilon$, so we define

$$D = \left\{(t, x) \in \mathbb{R}^2 : 0 \le t \le 2 - \epsilon \text{ and } 0 \le x \le \frac{2 - t}{2}\right\},$$

and for $(t, x) \in D$ we let

$$F(t, x) = -\frac{x}{2 - x - t}.$$

Notice that for $(t, x) \in D$,

$$2 - x - t \ge 2 - \frac{2 - t}{2} - t = \frac{2 - t}{2} > 0, \tag{11}$$

so $F(t, x)$ is defined.

By (9) and (10), any n-walk and the curve $x = f_x(t)$ both stay in the region D up to time $t = 2 - \epsilon$, and by (7), if $0 \le t \le 2 - \epsilon$ then $f_x'(t) = F(t, f_x(t))$. Thus, it makes sense to apply Lemma 5 to the functions F and f_x on the region D. In preparation for this, we make some observations about these functions. We first note that by (11) and the definition of D, for $(t, x) \in D$ we have

$$2 - x - t \ge \frac{2 - t}{2} \ge x \ge 0.$$

Since $F(t, x) = -x/(2 - x - t)$, it follows that

$$-1 \le F(t, x) \le 0, \tag{12}$$

and therefore

$$|f_x'(t)| = |F(t, f_x(t))| \le 1. \tag{13}$$

Next we compute

$$\frac{\partial F}{\partial x}(t,x) = -\frac{2-t}{(2-x-t)^2}, \qquad f_x''(t) = \frac{f_x(t)^2}{(2-f_x(t)-t)^3} = \frac{(F(t,f_x(t)))^2}{2-f_x(t)-t}.$$

Thus, if $(t,x) \in D$, then by (11),

$$\left|\frac{\partial F}{\partial x}(t,x)\right| = \frac{2-t}{(2-x-t)^2} \le \frac{2-t}{((2-t)/2)^2} = \frac{4}{2-t} \le \frac{4}{\epsilon}.$$

Similarly, if $0 \le t \le 2-\epsilon$, then

$$|f_x''(t)| = \frac{(F(t,f_x(t)))^2}{2-f_x(t)-t} \le \frac{1}{2-f_x(t)-t} \le \frac{1}{(2-t)/2} = \frac{2}{2-t} \le \frac{2}{\epsilon}.$$

We can therefore use $C_1 = 4/\epsilon$ and $C_2 = 2/\epsilon$ in Lemma 5. For reasons that will become clear later, the value we will use for δ in Lemma 5 is

$$\delta = \frac{C_1 \epsilon}{6(e^{2C_1}-1)}. \tag{14}$$

Since the function $F(t,x)$ is uniformly continuous on D, we can choose some $\zeta > 0$ such that for any two points $(t_1,x_1),(t_2,x_2) \in D$,

$$\text{if } |t_1 - t_2| < \zeta \text{ and } |x_1 - x_2| < \zeta \text{ then } |F(t_1,x_1) - F(t_2,x_2)| < \frac{\delta}{4}. \tag{15}$$

We now choose a positive integer m large enough that

$$\frac{2}{m} < \frac{\epsilon}{3}, \qquad \frac{2}{m} < \zeta, \qquad \frac{e^{2C_1}-1}{2m} < \frac{\epsilon}{6}. \tag{16}$$

Again, the reason for this choice will become clear later.

Consider an n-walk for any $n \ge m^2$. As in the statement of Lemma 4, let the x coordinates of the points on the walk be $x_0 = 1,\ x_1,\ \ldots,\ x_{2n} = 0$, and for $0 \le i \le 2n$ let $t_i = i/n$. We now divide $2n$ by m, getting a quotient q and remainder r. In other words,

$$2n = mq + r$$

and $0 \le r < m$. Notice that since $n \ge m^2$, we have $q \ge 2m$. We think of the walk as consisting of m blocks of steps, with each block containing q steps, followed by r extra steps at the end. For $0 \le j \le m$, let (T_j, X_j) be the position of the walk after j blocks of steps have been traversed. Thus, $T_j = t_{jq} = jq/n$ and $X_j = x_{jq}$.

Let $h = q/n$, so that for $0 \le j < m$,

$$T_{j+1} - T_j = h,$$

and note that since x either remains fixed or decreases by $1/n$ in each step of the walk,

$$0 \le X_j - X_{j+1} \le \frac{q}{n} = h.$$

Applying (16), we see that

$$h = \frac{2q}{2n} = \frac{2q}{mq + r} \le \frac{2q}{mq} = \frac{2}{m} < \frac{\epsilon}{3},$$

so

$$|T_{j+1} - T_j| \le \frac{2}{m} < \frac{\epsilon}{3}, \qquad |X_{j+1} - X_j| \le \frac{2}{m} < \frac{\epsilon}{3}. \tag{17}$$

In other words, in the course of a single block of steps, x and t change by less than $\epsilon/3$.

For $0 \le j < m$, let

$$\delta_j = \frac{X_{j+1} - X_j}{h} - F(T_j, X_j).$$

Rearranging this definition, this means that

$$X_{j+1} = X_j + h(F(T_j, X_j) + \delta_j).$$

Of course, this is the recurrence in our modified version of Euler's method.

We would now like to apply Lemma 5, but we have no guarantee that δ will be a bound on the numbers $|\delta_j|$. However, we can show that *if* δ is such a bound, then the walk stays in the ϵ-corridor:

Claim. *Suppose that for all $j < m$, if $T_j \le 2 - 2\epsilon$ then $|\delta_j| \le \delta$. Then the n-walk stays inside the ϵ-corridor.*

Proof of Claim. Notice that since $q \ge 2m$ and $2/m < \epsilon/3$,

$$T_m = t_{mq} = \frac{mq}{n} = \frac{2mq}{2n} = \frac{2mq}{mq + r} > \frac{2mq}{m(q+1)} = 2 - \frac{2}{q+1} > 2 - \frac{2}{2m} > 2 - \frac{\epsilon}{6} > 2 - 2\epsilon.$$

Thus, we can let k be the least index such that $T_k > 2 - 2\epsilon$. Then for all $j < k$, $T_j \le 2 - 2\epsilon$, and therefore, by assumption, $|\delta_j| \le \delta$. And since $T_{k-1} \le 2 - 2\epsilon$, by (17) we have

$$T_k < T_{k-1} + \frac{\epsilon}{3} \le 2 - 2\epsilon + \frac{\epsilon}{3} < 2 - \epsilon.$$

We can therefore apply Lemma 5 to the points (T_j, X_j) for $0 \le j \le k$ and the functions F and f_x on the region D to conclude that for all such j,

$$|X_j - f_x(T_j)| \le \left(\frac{hC_2}{2C_1} + \frac{\delta}{C_1}\right)\left((1 + C_1 h)^j - 1\right).$$

Since $j \le k \le m$ and $h \le 2/m$,

$$(1 + C_1 h)^j \le \left(1 + \frac{2C_1}{m}\right)^m < e^{2C_1},$$

where the last inequality is well known (see, for example, inequality 4.5.13 in [7]). Therefore

$$|X_j - f_x(T_j)| < \left(\frac{(2/m)(2/\epsilon)}{2(4/\epsilon)} + \frac{\delta}{C_1}\right)(e^{2C_1} - 1) = \frac{e^{2C_1} - 1}{2m} + \frac{\delta(e^{2C_1} - 1)}{C_1}.$$

By (16) and (14), the last two fractions are both at most $\epsilon/6$. Thus, we have shown that

$$|X_j - f_x(T_j)| < \frac{\epsilon}{3}. \tag{18}$$

This implies that all of the points (T_j, X_j) for $0 \le j \le k$ are in the ϵ-corridor.

Since $T_k > 2-2\epsilon$, as we observed after (10), all points on the n-walk beyond (T_k, X_k) are also in the ϵ-corridor. We still need to worry about points on the n-walk in the interiors of the first k blocks. If (t, x) is such a point, then (t, x) occurs between (T_j, X_j) and (T_{j+1}, X_{j+1}), for some $j < k$. To see that (t, x) is in the ϵ-corridor, we compute

$$|x - f_x(t)| \le |x - X_j| + |X_j - f_x(T_j)| + |f_x(T_j) - f_x(t)|.$$

We now bound each of the terms on the right-hand side. We already know, by (17) and (18), that $|x - X_j| \le |X_{j+1} - X_j| < \epsilon/3$ and $|X_j - f_x(T_j)| < \epsilon/3$. For the third term we apply the mean value theorem:

$$f_x(T_j) - f_x(t) = f'_x(c)(T_j - t),$$

for some c between t and T_j. By (13) and (17), we conclude that

$$|f_x(T_j) - f_x(t)| = |f'_x(c)| \cdot |T_j - t| \le |f'_x(c)| \cdot |T_{j+1} - T_j| < 1 \cdot \frac{\epsilon}{3} = \frac{\epsilon}{3}.$$

Putting it all together, we get

$$|x - f_x(t)| \le |x - X_j| + |X_j - f_x(T_j)| + |f_x(T_j) - f_x(t)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

so the point (t, x) is in the ϵ-corridor. We have now shown that all points on the walk are in the ϵ-corridor, which completes the proof of the claim. □

The claim shows that if an n-walk goes outside of the ϵ-corridor, then there must be some $j < m$ such that $T_j \le 2 - 2\epsilon$ and $|\delta_j| > \delta$. To complete the proof, we will show that this is unlikely to happen.

Partition $\{(t, x) \in D : t \le 2 - 2\epsilon\}$ into finitely many disjoint regions R_1, R_2, …, R_K, each with diameter less than ζ. By (12) and (15), for each k with $1 \le k \le K$ we can choose a number r_k such that $-1 \le r_k \le 0$ and for every $(t, x) \in R_k$,

$$|F(t, x) - r_k| < \frac{\delta}{4}. \tag{19}$$

For example, we can take r_k to be $F(t, x)$ for some particular $(t, x) \in R_k$. Notice that the regions R_k and numbers r_k do not depend on n; as $n \to \infty$, R_k and r_k will remain fixed.

We will write $\Pr_n(E)$ to denote the probability that an event E occurs when an n-walk takes place. The claim implies that the probability that an n-walk will leave the ϵ-corridor is at most

$$\sum_{j=0}^{m-1} \sum_{k=1}^{K} p_{j,k}(n),$$

where

$$p_{j,k}(n) = \Pr_n((T_j, X_j) \in R_k \text{ and } |\delta_j| > \delta).$$

Thus, it will suffice to show that for each j and k, $\lim_{n\to\infty} p_{j,k}(n) = 0$.

Fix j and k with $0 \le j < m$ and $1 \le k \le K$. The value of δ_j is determined by the block of steps taken by the n-walk in going from (T_j, X_j) to (T_{j+1}, X_{j+1}). The points on this part of the walk are (t_{jq+i}, x_{jq+i}) for $0 \le i \le q$. We will refer to the step from (t_{jq+i}, x_{jq+i}) to (t_{jq+i+1}, x_{jq+i+1}) as *step* i of this block of the n-walk. Notice that there are q steps in the block, and since q is the quotient when n is divided by m and m is fixed, $q \to \infty$ when $n \to \infty$.

Let a be the number of steps in the block in which x decreases by $1/n$. In the remaining $q - a$ steps the value of x does not change, so $X_j - X_{j+1} = a/n$. Therefore, by definition,

$$\delta_j = \frac{X_{j+1} - X_j}{h} - F(T_j, X_j) = -\frac{a/n}{q/n} - F(T_j, X_j) = -\frac{a}{q} - F(T_j, X_j).$$

Although the value of $p_{j,k}(n)$ does not depend on the precise method by which the steps in this block of the walk are chosen, it will be helpful to specify a method. We will assume that for $0 \le i < q$, random numbers s_i are chosen, independently and uniformly in $[0, 1]$, and then in step i, x decreases by $1/n$ if

$$s_i < \frac{x_{jq+i}}{2 - x_{jq+i} - t_{jq+i}} = -F(t_{jq+i}, x_{jq+i}),$$

and x is unchanged otherwise. Of course, according to equation (6), this procedure generates the correct probabilities for the steps of the walk.

Suppose that $(T_j, X_j) \in R_k$. Then by (19), $|F(T_j, X_j) - r_k| < \delta/4$, or in other words

$$-r_k - \frac{\delta}{4} < -F(T_j, X_j) < -r_k + \frac{\delta}{4}. \tag{20}$$

Also, for $0 \le i < q$, by (17) and (16), $|t_{jq+i} - T_j| \le 2/m$, $|x_{jq+i} - X_j| \le 2/m$, $2/m < \epsilon/3$, and $2/m < \zeta$. Since $t_{jq+i} \le T_j + 2/m < 2 - 2\epsilon + \epsilon/3 < 2 - \epsilon$, we have $(t_{jq+i}, x_{jq+i}) \in D$, and therefore, by (15), $|F(t_{jq+i}, x_{jq+i}) - F(T_j, X_j)| < \delta/4$. Combining this with $|F(T_j, X_j) - r_k| < \delta/4$, we conclude that $|F(t_{jq+i}, x_{jq+i}) - r_k| < \delta/2$, or in other words

$$-r_k - \frac{\delta}{2} < -F(t_{jq+i}, x_{jq+i}) < -r_k + \frac{\delta}{2}.$$

Recall that step i is determined by how s_i compares to $-F(t_{jq+i}, x_{jq+i})$. We can now draw the conclusion that if $(T_j, X_j) \in R_k$, then:

(a) if $s_i \le -r_k - \dfrac{\delta}{2}$, then at step i, x decreases by $\dfrac{1}{n}$;

(b) if $s_i \ge -r_k + \dfrac{\delta}{2}$, then at step i, x remains unchanged.

We are now ready to show that $\lim_{n\to\infty} p_{j,k}(n) = 0$. By definition,

$$p_{j,k}(n) = \Pr_n((T_j, X_j) \in R_k \text{ and } \delta_j > \delta) + \Pr_n((T_j, X_j) \in R_k \text{ and } \delta_j < -\delta).$$

We will show that both of the probabilities on the right-hand side approach 0 as $n \to \infty$.

For the first, suppose that $(T_j, X_j) \in R_k$ and $\delta_j > \delta$. Since $\delta_j = -a/q - F(T_j, X_j)$, by (20) this implies that

$$\frac{a}{q} < -F(T_j, X_j) - \delta < -r_k - \frac{3\delta}{4}.$$

Now let a' be the number of values of i for which $s_i \le -r_k - \delta/2$. By conclusion (a) above, $a' \le a$, and therefore

$$0 \le \frac{a'}{q} \le \frac{a}{q} < -r_k - \frac{3\delta}{4} < -r_k - \frac{\delta}{2} < 1.$$

This is very unlikely to happen. To see why, notice first that for $0 \le i < q$, since s_i is chosen uniformly in $[0, 1]$ and $0 < -r_k - \delta/2 < 1$, the probability that $s_i \le -r_k - \delta/2$ is $-r_k - \delta/2$. And since the s_i are chosen independently, this means that a'/q, which is the fraction of values of i for which $s_i \le -r_k - \delta/2$, should be close to $-r_k - \delta/2$. More precisely, by the law of large numbers (see [2, Section VI.4, p. 152]), for any $\alpha > 0$, the probability that $|a'/q - (-r_k - \delta/2)| > \alpha$ must approach 0 as $q \to \infty$. And since $q \to \infty$ as $n \to \infty$, taking $\alpha = \delta/4$ we can conclude that

$$\lim_{n\to\infty} \Pr_n\left(\frac{a'}{q} < -r_k - \frac{3\delta}{4}\right) = 0.$$

It follows that

$$\lim_{n\to\infty} \Pr_n((T_j, X_j) \in R_k \text{ and } \delta_j > \delta) = 0.$$

The second probability is similar. If $(T_j, X_j) \in R_k$ and $\delta_j < -\delta$, then

$$\frac{a}{q} > -F(T_j, X_j) + \delta > -r_k + \frac{3\delta}{4}.$$

Now let a' be the number of values of i for which $s_i < -r_k + \delta/2$. This time we use fact (b) above to conclude that $a' \ge a$, so

$$1 \ge \frac{a'}{q} \ge \frac{a}{q} > -r_k + \frac{3\delta}{4} > -r_k + \frac{\delta}{2} > 0.$$

Once again, the law of large numbers says that the probability of this event goes to 0 as $n \to \infty$, which completes the proof of Lemma 4 and, therefore, Theorem 1.

4 Proofs of Theorems 2 and 3

To prove Theorem 2, fix $n > 0$, and let A denote the number of whole pills removed from the bottle before the first half pill. Of course, the first pill removed from the bottle must be a whole pill, and there are n whole pills altogether, so $1 \le A \le n$.

For $1 \le k \le n$, let $X_k = 1$ if the first k pills removed from the bottle are all whole pills, and $X_k = 0$ otherwise. Then we have $A = X_1 + X_2 + \cdots + X_n$, and therefore

$$E(A) = E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n).$$

The probability that the first pill removed is a whole pill is 1. Once the first whole pill has been removed, the bottle contains $n-1$ whole pills and 1 half pill, so the probability that the second pill is also a whole pill is $(n-1)/n$. Similarly, if the first two pills are whole pills, then the probability that the third pill is a whole pill is $(n-2)/n$. Continuing in this way, we see that for $1 \le k \le n$,

$$\begin{aligned} E(X_k) &= \Pr(X_k = 1) \\ &= 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n} \\ &= \frac{n!}{n^k (n-k)!}. \end{aligned}$$

Thus,

$$E(A) = \sum_{k=1}^{n} E(X_k) = \sum_{k=1}^{n} \frac{n!}{n^k (n-k)!}.$$

Reindexing by $j = n-k$, we get

$$E(A) = \sum_{k=1}^{n} \frac{n!}{n^k (n-k)!} = \sum_{j=0}^{n-1} \frac{n!}{n^{n-j} j!} = \frac{n!}{n^n} \sum_{j=0}^{n-1} \frac{n^j}{j!}. \tag{21}$$

To relate this formula to the incomplete gamma function, we first evaluate the integral in the definition of the incomplete gamma function. Applying integration by parts k times leads to the formula in the following lemma.

Lemma 6. *For every integer $k \ge 0$,*

$$\int t^k e^{-t}\, dt = -\frac{k!}{e^t} \sum_{j=0}^{k} \frac{t^j}{j!} + C.$$

Using this lemma, we find that

$$\Gamma(n,n) = \int_n^\infty t^{n-1} e^{-t}\, dt = \lim_{N \to \infty} \left[-\frac{(n-1)!}{e^t} \sum_{j=0}^{n-1} \frac{t^j}{j!} \right]_n^N = \frac{(n-1)!}{e^n} \sum_{j=0}^{n-1} \frac{n^j}{j!}. \tag{22}$$

Thus,

$$\sum_{j=0}^{n-1} \frac{n^j}{j!} = \frac{e^n}{(n-1)!} \Gamma(n,n).$$

Substituting into (21), we get

$$E(A) = \frac{n!}{n^n} \sum_{j=0}^{n-1} \frac{n^j}{j!} = \frac{n!}{n^n} \cdot \frac{e^n}{(n-1)!} \Gamma(n,n) = \frac{e^n}{n^{n-1}} \Gamma(n,n).$$

This proves the first statement in Theorem 2.

To prove the second statement, about the asymptotic value as $n \to \infty$, we need the following fact.

Lemma 7.

$$\lim_{n\to\infty} \frac{\Gamma(n,n)}{(n-1)!} = \frac{1}{2}.$$

Proof. According to inequality 8.10.13 of [7],

$$\frac{\Gamma(n,n)}{(n-1)!} < \frac{1}{2} < \frac{\Gamma(n+1,n)}{n!}. \tag{23}$$

By Lemma 6 and equation (22),

$$\Gamma(n+1,n) = \int_n^\infty t^n e^{-t}\,dt = \frac{n!}{e^n}\sum_{j=0}^{n}\frac{n^j}{j!} = n\frac{(n-1)!}{e^n}\sum_{j=0}^{n-1}\frac{n^j}{j!} + \frac{n^n}{e^n} = n\Gamma(n,n) + \frac{n^n}{e^n}.$$

Substituting into the second half of inequality (23), we get

$$\frac{1}{2} < \frac{\Gamma(n,n)}{(n-1)!} + \frac{n^n}{e^n n!},$$

and therefore

$$\frac{1}{2} - \frac{n^n\sqrt{2\pi n}}{e^n n!}\cdot\frac{1}{\sqrt{2\pi n}} < \frac{\Gamma(n,n)}{(n-1)!} < \frac{1}{2}.$$

By Stirling's formula, $\lim_{n\to\infty} n^n\sqrt{2\pi n}/(e^n n!) = 1$, and the lemma now follows by the squeeze theorem. □

This lemma allows us to determine the asymptotic rate of growth of the expected value of A. The expected length of the initial run of whole pills can be rewritten in the form

$$E(A) = \frac{e^n}{n^{n-1}}\Gamma(n,n) = \sqrt{2\pi n}\cdot\frac{e^n n!}{n^n\sqrt{2\pi n}}\cdot\frac{\Gamma(n,n)}{(n-1)!} \sim \sqrt{2\pi n}\cdot 1\cdot\frac{1}{2} = \sqrt{\frac{\pi n}{2}},$$

which completes the proof of Theorem 2.

Finally, we give Stong's proof of Theorem 3. For $1 \le k \le n$, consider the kth whole pill that is removed from the bottle. This pill is cut in half, and half of it is returned to the bottle; we will refer to this half pill as the *kth half pill*. Let $X_k = 1$ if the kth half pill is removed from the bottle after the last whole pill is removed, and $X_k = 0$ otherwise. Then the expected value we seek is

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n).$$

After the kth half pill has been returned to the bottle, there are $n-k$ whole pills still in the bottle, and we have $X_k = 1$ if and only if among the set of pills consisting of these $n-k$ remaining whole pills and the kth half pill, the half pill is the last one to be removed from the bottle. Since each pill in this set is equally likely to be chosen at each step, we have

$$E(X_k) = \Pr_n(X_k = 1) = \frac{1}{n-k+1}.$$

Therefore the expected number of half pills removed from the bottle after the last whole pill is

$$E(X_1) + E(X_2) + \cdots + E(X_n) = \frac{1}{n} + \frac{1}{n-1} + \cdots + 1 = H_n.$$

5 Variations

In all of our calculations, we have assumed that when a pill is removed from the bottle, all pills in the bottle are equally likely to be chosen. But since the whole pills are twice as big as the half pills, another natural assumption would be that whole pills are twice as likely to be chosen as half pills. In this section we summarize the results of redoing our calculations with this alternative assumption, leaving the details to the reader.

If whole pills are twice as likely to be chosen as half pills, then the differential equations (1) must be replaced by

$$\frac{dx}{dt} = -\frac{2x}{2x+y}, \qquad \frac{dy}{dt} = \frac{2x-y}{2x+y}.$$

The solution to this system of equations that passes through the point $(1,0)$ is

$$y = 2(\sqrt{x} - x), \qquad x = \frac{(2-t)^2}{4}, \qquad y = \frac{t(2-t)}{2}.$$

Once again, the random walk converges uniformly in probability to this curve as $n \to \infty$.

Surprisingly, in this case the expected number of whole pills removed before the first half pill turns out to be exactly the same as the expected number of half pills removed after the last whole pill. Calculations similar to those in the last section show that both expected values are

$$\frac{2^{2n}}{\binom{2n}{n}} - 1.$$

There is a simple explanation for why these two expected values are equal. The explanation is based on an alternative procedure we could follow to decide which pill to remove from the bottle each day. First, number the pills in a full bottle from 1 to n. Then make a deck of $2n$ cards numbered from 1 to n, with each number appearing on two cards, and shuffle the deck. Every day, deal a card from the top of the deck, and if the card has the number k on it, then remove pill number k from the bottle. As usual, if the pill is whole, then cut it in half and return half to the bottle.

On any day, if pill number k is still whole, then there will be two cards numbered k in the deck; if half of pill number k has already been taken, then there will be only one card numbered k in the deck; and if pill number k has been used up completely, then there will be no cards numbered k left in the deck. It follows that whole pills will be twice as likely to be chosen as half pills, as required.

If we follow this procedure, then the number of whole pills removed from the bottle before the first half pill is removed will be the same as the number of distinct cards dealt from the top of the deck before the first duplicate card. Similarly, we could determine how many half pills will be removed from the bottle after the last whole pill by dealing cards from the bottom of the deck and counting the number of distinct cards dealt before the first duplicate. It should now be clear by symmetry that the expected values of these two numbers are equal. Indeed, the problem of computing this common expected value is equivalent to the third question addressed in [9], and the answer follows from Theorem 5 of [9].

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